Sign reversal of the boson-boson interaction potential for the planar Bose-Fermi mixtures under synthetic magnetic field

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We study the mutually coupled, strongly interacting bosonic and non-interacting fermionic, species of unequal masses in the regime were the retardation effects are an important part of the physics. A cloud of neutral atoms experiences a synthetic magnetic field because of a vector potential that imposes a phase shift on the constituents. The magnetic field causes the oscillations of the magnitude and sign of the effective interaction between bosons from repulsive to attractive in contrast to the static case. We show that the dynamics for the gaseous Bose-Fermi mixtures when reaching the quantum-Hall regime becomes highly nontrivial.

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The ultracold gases of atoms allow for the observation and control of many-body quantum effects at macroscopic scales. They can thus offer a fertile arena for exploration of condensed matter physics. The obvious limitations that come from neutrality of atoms preclude the observation of great variety of fundamental phenomena i.e. charge particle moving in magnetic field. However, the analysis of the latter can be carried considering rotating Bose-Einstein condensates [1–3] trapped in lattice potential created by lasers. In a frame of reference rotating about the z-axis with angular velocity Ω the kinetic term in Hamiltonian is equivalent to that of a particle of charge Q experiencing a magnetic field B with $QB = 2m\Omega$, where m is the mass of the particle [4]. This connection shows that the Coriolis force in the rotating frame plays the same role as the Lorentz force on a charged particle in an uniform magnetic field [5, 6]. The above setting comes with limitations because a large magnetic fields $f \equiv ma^2\Omega/\pi\hbar$ (angular momentum) are required to make possible the study of poorly explored bosonic states in case when $f \equiv p/q$, (p and q is the ratio of atom number to the number of flux quanta respectively) is a rational number. Very recently Lin *et al.* [7] circumvented the problem by imprinting a quantum mechanical phase on the neutral atoms without rotating them at all and the quantum-Hall physics can be reached.

Current experiments [8–12] on trapped mixtures of the atomic Bose-Fermi (BF) and Bose-Bose gases show that the presence of a relevant fraction of one modifies the quantum phase transition occurring in the other inducing a significant loss of coherence. These observations are supported by a theoretical description that includes the multiband virtual transitions [13], different masses of strongly interacting particles [14] and numerical calculations [15]. The density-density (DD) interaction between different species can be repulsive or attractive and is produced by changes of one species density that induce a modulation of another. Therefore the dynamics underlying the phase transitions in the BF mixtures is produced by the feedback of the density perturbation and a shift of the inter-bosonic potential occurs, that changes the original interaction between them [16] providing various novel phases [17].

In the present paper, motivated by recent experiments done by Lin *et al.* [7], we calculated the form of the effective inter-bosonic potential when a synthetic magnetic field (SMF) is applied to neutral gaseous Bose-Fermi mixtures. We predict that the fermion-mediated effective interaction between bosons has a complicated pattern of the frequency dependent magnitude. Moreover, the SMF renders the inter-bosonic potential oscillatory with sign change, thus switching it between repulsive and attractive. As a consequence the resonances appear and BF mixture that enters the quantum-Hall regime displays surprisingly complex dynamics unreachable in conventional solid state physics. We expect that our theoretical results open up the experimental studies [18] of the renormalized interaction energies in stable many-body phases with strong correlations and their dynamical properties.

Restricting our analysis to the lowest energy band of a square optical lattice in synthetic magnetic field, the Bose-Fermi quantum gaseous mixture can be modeled via the following Hamiltonian [19]:

$$\begin{aligned}\mathcal{H} &= \frac{U_b}{2}\sum_i n_{bi}\left(n_{bi}-1\right) - \sum_{\langle i,j\rangle} t_{bij} b_i^\dagger b_j - \mu_b \sum_i n_{bi} \\ &- \sum_{\langle i,j\rangle} t_{fij} c_i^\dagger c_j - \mu_f \sum_i n_{fi} + U_{bf}\sum_i n_{bi} n_{fi}, \qquad (1)\end{aligned}$$

where $b_i^\dagger (c_i^\dagger)$ and $b_j (c_j)$ stand for the bosonic (fermionic) creation and annihilation operators; $n_{bi} = b_i^\dagger b_i (n_{fi} = c_i^\dagger c_i)$ measures the corresponding boson (fermion) number on the site i, $U_b > 0$ is the on-site repulsion and $\mu_b(\mu_f)$ stands for the chemical potential for bosons (fermions). The DD interaction between the bosonic and non-interacting, spin-polarized (collisions in the s-wave channel are forbidden by their statistics), fermionic atoms is denoted by U_{bf} and depends, on boson to fermion mass ratio m_b/m_f. Here, $\langle i, j\rangle$ identifies summation over the nearest-neighbor sites. Furthermore $t_b(t_f)$

sets the kinetic energy scale for bosons (fermions).

A synthetic magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}(\boldsymbol{r})$ enters the Hamiltonian Eq. (1) through the Peierls phase factor according to $t_{ij} \rightarrow t_{ij} \exp\left(\frac{2\pi i}{\Phi_0}\int_{\boldsymbol{r}_j}^{\boldsymbol{r}_i} \mathbf{A}\cdot d\boldsymbol{l}\right)$, where $\Phi_0 = hc/e$ is the flux quantum and e is an elementary charge. Thus, the phase shift on each site is determined by the vector potential $\mathbf{A}(\boldsymbol{r})$ and can be controlled experimentally [7]. The magnetic field is introduced in the theory by the density of states (DOS). There are significant difficulties in obtaining and analyzing and analyze the solutions of the above analytically for every value of f. Only a few closed formulas for DOS are accessible [20] and consequently not every applied magnetic field can be described theoretically. We expanded the set of the available analytical solutions obtaining closed formulas also for $f = 1/8$ and $3/8$. This allows us the detailed analysis of the dynamical response function which have been found to play a crucial role in complex systems. The partition function of bosonic and fermionic mixtures is written in the form $\mathcal{Z} = \int\left[\mathcal{D}\bar{b}\mathcal{D}b\mathcal{D}\bar{c}\mathcal{D}c\right] e^{-\mathcal{S}[b,c]}$ with action given by

$$\mathcal{S} = \mathcal{S}_b + \mathcal{S}_c + \int_0^\beta d\tau \mathcal{H}(\tau) \tag{2}$$

where $\mathcal{S}_b = \sum_i \int_0^\beta d\tau \bar{b}_i \frac{\partial}{\partial\tau} b_i$ and $\mathcal{S}_c = \sum_i \int_0^\beta d\tau \bar{c}_i \frac{\partial}{\partial\tau} c_i$. Using the bosonic (fermionic) path integral over the complex fields depending on the "imaginary time" $0 \le \tau \le \beta \equiv 1/k_{\mathrm{B}}T$ with T being the temperature we can easily integrate over the fermionic fields [14] because spins are frozen due to influence of the magnetic trap and there is no direct interaction between fermions. After that, we obtain the partition function in the form $\mathcal{Z} = \int\left[\mathcal{D}\bar{b}\mathcal{D}b\mathcal{D}\bar{c}\mathcal{D}c\right] e^{-\mathcal{S}_b[b,n_b]} e^{-\mathrm{Tr}\ln\hat{G}_c}$. The trace of the two-point correlation function for noninteracting fermions $\hat{G}_c$, after exploiting Fourier-Matsubara transform reads:

$$\mathrm{Tr}\ln\hat{G}_c = -\frac{U_{bf}^2}{2}\sum_{\boldsymbol{k},\ell} \Lambda_{\boldsymbol{k}}(\omega_\ell)\,\chi_{\boldsymbol{k}}(i\nu_\ell)\,\Lambda_{-\boldsymbol{k}}(-\omega_\ell), \tag{3}$$

where $\omega_\ell = 2\pi\ell/\beta$ ($\nu_\ell = \pi(2\ell+1)/\beta$) with $\ell = 0, \pm1, \pm2, ...$ are the Bose(Fermi)-Matsubara frequencies respecting periodic (antiperiodic) boundary conditions of the bosonic (fermionic) field operator with $\Lambda_{\boldsymbol{k}}(\omega_\ell) = \bar{b}_{\boldsymbol{k}}(\omega_\ell)\, b_{\boldsymbol{k}}(\omega_\ell)$ and

$$\chi_{\boldsymbol{k}}(i\nu_\ell) = \sum_{\mathbf{k}'} \frac{n_{\mathrm{F}}\left(t_{f\boldsymbol{k}'}^{p/q}\right) - n_{\mathrm{F}}\left(t_{f\boldsymbol{k}'+\boldsymbol{k}}^{p/q}\right)}{t_{f\boldsymbol{k}'}^{p/q} - t_{f\boldsymbol{k}'+\boldsymbol{k}}^{p/q} - i\nu_\ell}, \tag{4}$$

is the Lindhard function - more commonly called the random phase approximation with $n_{\mathrm{F}}(x)$ being the Fermi distribution; $t_{\boldsymbol{k}'}^{p/q}$ is the dispersion relation calculated from Harper equation [20]. It correctly predicts a number of properties of the collective phenomena in electron gas such as plasmons [21]. To stay in the local regime we perform $\boldsymbol{k}$ and $\boldsymbol{k}'$ integration over the first Brillouin zone and, in the $T \rightarrow 0$ limit, using an analytic continuation, we obtain imaginary part $\chi''(\omega)$ of the local dynamic Lindhard function (LDLF). Therefore, the corresponding real part $\chi'(\omega)$ can be deduced from the Kramers-Krönig relation. We noticed that $\chi''(\omega)$ is proportional to the absorption spectrum of the medium so it can be directly measured. Now, doing the inverse Fourier transform Eq. 3 and using gradient expansion, we obtain quadratic form of the trace with extracted frequency dependence

$$\mathrm{Tr}\ln\hat{G}_c \rightarrow -\frac{U_{bf}^2\chi'(\omega)}{2}\sum_i \int_0^\beta d\tau \left[\bar{b}_i(\tau)\, b_i(\tau)\right]^2. \tag{5}$$

The consequence of the difference in masses of bosons and fermions is the fact that the speed of the Bogoliubov sound v_b for bosons differs from the first sound v_f of the ideal Fermi gas. In typical experimental realizations the acoustic long-wavelength boson and fermion velocities are comparable and both constituents equilibrates similarly. The mentioned different mass ratio has far-reaching consequences, including the possibility of generating the DD oscillations [22]. With these concerns in mind we do not restrict our calculations to the static limit but consider also the local dynamical response function hence the retardation effects are an important part of the physics. Moreover, we see [14] from the $U_{bf}(m_b/m_f)$ dependence that even if the fermion mass is very large system has a finite value of the interaction strength. On the other hand slow bosonic atoms would affect the system providing strong inter-species coupling considerably faster. We neglected the back action of the bosons on the fermions (renormalization of the fermion DD correlation function) because of the diluteness of the system and theory limitations condition $U_b > U_{bf}^2\chi'(\omega)$ [23].

Our analysis is carried out in the frequency domain, although measurements are sometimes made in the time domain and then Fourier transformed to the frequency. When we add Eq. (5) to the bosonic part of the action there is a striking resemblance to the one-component Bose-Hubbard action with the original repulsive interaction replaced now by $U_b \rightarrow U_{\mathrm{eff}} = U_b + U_{bf}^2\chi'(\omega)$ which is the induced, frequency-dependent, effective inter-bosonic potential. From the above we see the DD correlations between the constituents give rise to additional interaction among bosons, which is *robust* to repulsive or attractive nature of the inter-species interaction but not to the sign of the LDLF. Before carrying out an analysis of the phase diagrams it is a good idea to examine the structure of the response function graphically Fig. 1. Structure that will usually reflect, at least in part, the physical processes present in the real systems. The interactions caused by the DD correlations may change its sign even if the system has not the phase shift imprinted on it as a result of the collective excitations. For values $f = 0$

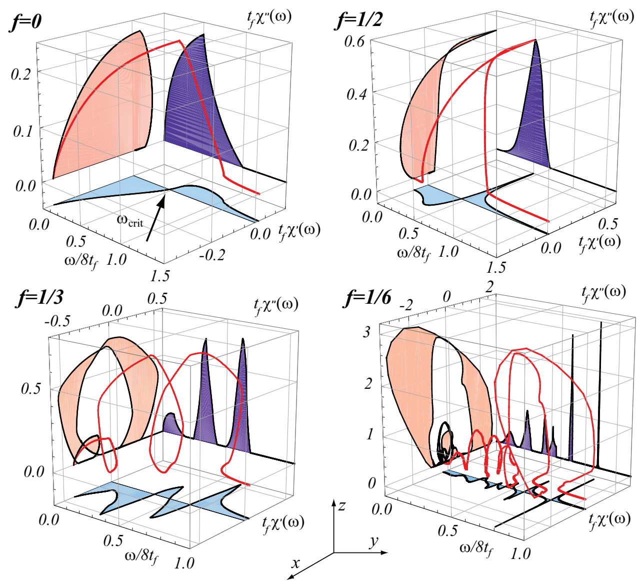

Figure 1: (Color online) Real $t_f\chi''(\omega)$ (x-y plane) and imaginary $t_f\chi'(\omega)$ (z-y plane) parts of the local, frequency $\omega/8t_f$ dependent, density-density response function in the synthetic magnetic field for several values of $f = 0$, $1/2$, $1/3$ and $1/6$ (for the case with $f = 1/4$ see supplemental material [24]). The complex plot (x-z plane) is the "Cole-Cole" like diagram [25]. The three-dimensional parametric curve shows the evolution of the density-density response function with the frequency and guarantees the stability of the system according to the Nyquist theorem (see text). Normalization here $t_f \equiv t_f^{p/q=0}$.

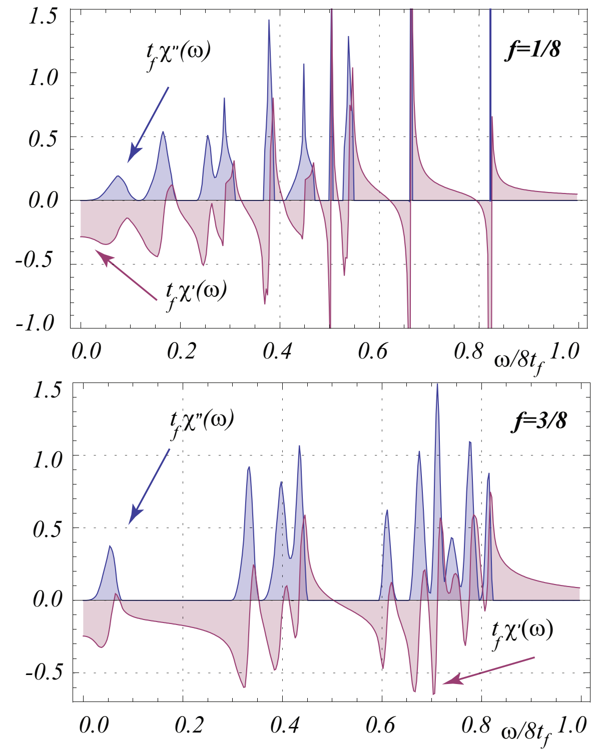

Figure 2: (Color online) Real $t_f\chi''(\omega)$ and imaginary $t_f\chi'(\omega)$ parts of the local, frequency $\omega/8t_f$ dependent, density-density response function in the synthetic magnetic field for $f = 1/8$ and $f = 3/8$. Normalization here $t_f \equiv t_f^{p/q=0}$.

and 1/2 we have only one resonance and induced attractive interaction between bosons becomes repulsive above some critical frequency $\omega_{\rm crit}$. Decreasing the strength of SMF makes the physics nontrivial and for $q > 2$ the number of the resonances increase significantly Fig. 2. Such complex behavior of the mixtures emerges in the limit *not reachable* in conventional systems of condensed matter physics because the very high values of magnetic field are required to acquire the desired range $f \leq 1/2$. The method of an optically synthesized magnetic field for ultracold neutral atoms [7] already reached that regime with a high accuracy. We see that the situation is greatly modified if one applies lower magnetic fields. The narrower peaks in frequency domain the longer time we need to observe the collective responses from the system. According to the above we notice that the BF mixtures has to be in a very high magnetic field to make oscillations experimentally observable and seen as a density modulation. The stability of the effects we analyze comes from the Nyquist theorem for the frequency dependent diagram Fig. 1. The parametric curve $\chi'(\omega)$ vs $\chi''(\omega)$ as a function of frequency and the ensuing effective interaction described by its behavior $U_{\rm eff}$ automatically yields a stability if the curve does not encircle the origin [26].

To obtain an equation of state we apply the quantum-rotor approach, that successfully casts an essential part of the physics of strongly interacting fermions [27–29] and bosons [30] exclusively, to BF mixtures. We incorporate fully our calculations to the phase fluctuations $b_i(\tau) = [a_0 + a_i'(\tau)]\, e^{i\phi_i(\tau)}$ governed by the gauge U(1) group and drop corrections to the amplitude $a_i'(\tau)$ of the order parameter $\Psi_{\rm B} \equiv \left\langle b_i(\tau)\, e^{i\phi_i(\tau)}\right\rangle = b_0\psi_{\rm B}$. The non-vanishing value of the $\Psi_{\rm B}$ signals a bosonic condensation [30]. A phase $\phi_i(\tau)$ of the many-body wave function might be arbitrary but correlations among the local phases of its constituents can bring unusual gauge structures. Now, the partition function $\mathcal{Z} = \int [\mathcal{D}\phi]\, e^{-\mathcal{S}_{\rm ph}[\phi]}$, with an effective action can be expressed in *phase-only* terms

$$\mathcal{S}_{\rm ph}[\phi] = \int_0^\beta d\tau \left\{ \sum_i \left[\frac{\dot{\phi}_i^2(\tau)}{2U_{\rm eff}} + \frac{\bar{\mu}_b}{iU_{\rm eff}}\dot{\phi}_i(\tau) \right] - \sum_{\langle i,j\rangle} J^{p/q} e^{i[\phi_i(\tau)-\phi_j(\tau)]} \right\}. \quad (6)$$

The phase stiffness coefficient given by $J^{p/q} = t_b^{p/q}\left(8t_b^{p/q} + \bar{\mu}_b - U_{bf}N_{\rm F}\right)/U_b$ describes the hopping matrix elements renormalized by the amplitude of the order parameter and $N_{\rm F}$ is the average number of fermions.

The critical line equation that separates the Mott in-

sulator - superfluid transition Fig. 3, details of similar derivation of the critical line equation are described in [14], will take simple form:

$$1=\int_{-\infty}^{+\infty}\frac{\rho^{p/q}\left(\xi\right)d\xi}{\sqrt{2\bar{\xi}\left(8\frac{t_b}{U_b}+\frac{\mu_b}{U_b}-\eta\right)\frac{1}{\alpha}\frac{t_b}{U_b}+\upsilon^2\left(\frac{1}{\alpha}\frac{\mu_b}{U_b}\right)}} \tag{7}$$

where $\bar{\xi}=\xi_{\max}^{p/q}-\xi$ and $\xi_{\max}^{p/q}$ is the maximum of the band spectrum. The renormalization parameters are defined as: $\alpha=1+\frac{U_{bf}^2}{U_b}\chi'\left(\omega\right)$ and $\eta=\frac{U_{bf}}{U_b}N_{\rm F}-1/2$. In Eq. (7) $\upsilon\left(\mu/U\right)=\mathrm{frac}\left(\mu/U\right)-1/2$, where $\mathrm{frac}\left(x\right)$ is the fractional part of the number. Because the higher values of the normalized chemical potential for the fermions μ_f/t_f decreases $\chi'\left(\omega\right)$ and $\chi''\left(\omega\right)$ [14], consequently terms containing explicitly the average density of fermions $N_{\rm F}$ will acquire more significance than these with exclusively the inter-species interaction U_{bf}. The periodicity of the phase diagram can be easily characterized by its evolution with changing the α parameter and applied magnetic field (see Fig. 3). For $f=0$, taking $\omega<\omega_{\rm crit}$, we recover the previous theoretical results in which, after adding fermions to the system, the effective interaction $U_{\rm eff}\left(\omega\right)$ becomes smaller than repulsive energy U_b for bosons only and superfluid phase increases. However, in the local dynamic limit, when $\omega>\omega_{\rm crit}$ the Mott insulator phase becomes stronger and bosons tend to localize on the lattice sites. Imprinting a phase factor on neutral particles $f\neq 0$ can provide very complex behavior of the mixtures and induced effective interaction $U_{\rm eff}$ *can oscillate* from attractive to repulsive depending on frequency. The measurements by the Bragg spectroscopy method of the linear response of correlated two-dimensional BF mixtures at non-zero momentum transfer can give us more insight into the excitations spectra modified by the applied synthetic magnetic field.

In conclusion, we have studied a planar mixture of bosons and spinless fermions with synthetic magnetic field imposed on the system. We found that the underlying dynamics of mixture of particles with different statistics and masses entering a quantum-Hall regime is very complex allowing effective bosonic interaction to be switched between repulsive and attractive. The experimental evidence of our findings is feasible however precise measurements of the magnetic field are requisite which is possible with the recently developed optically synthesized magnetic field for neutral atoms [7].

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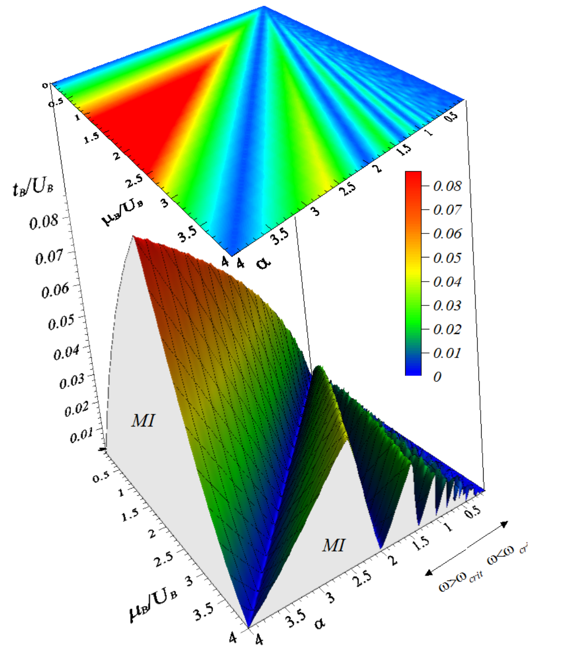

Figure 3: (Color online) The phase diagram showing the Mott insulator (MI) - superfluid transition (MI phase inside the lobes, superfluid above the surface) obtained from Eq. 7 of the Bose-Fermi mixtures confined in two-dimensional square optical lattice with synthetic magnetic field $f=1/2$ in the space of the parameters t_b/U_b-μ_b/U_b-α for $\eta=-2$ ($\omega>\omega_{\rm crit}$ means that system is in the repulsive regime $U_{\rm eff}>U_b$ with $\chi'\left(\omega\right)>0$, even though the interspecies interaction is negative $U_{bf}<0$, see also Fig. 1). Upper panel is the density plot of the three-dimensional phase diagram.

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